2.11 The Branch-and-Bound Algorithm

Enumeration is the most obvious way to solve integer programmes: enumerate all feasible solutions (since in most cases their number is finite) to find the best possible objective value. However, except in very small instances, total enumeration (or exhaustive search) is hopeless.

In practice,

by solving a single subproblem, many possible solutions may be eliminated from consideration.

Branch-and-Bound is a method which implicitly enumerate all possible solutions to an IP, and thus also known as a partial enumeration method. It requires the usage of lower and upper bounds that we have discussed in the LP relaxation section:

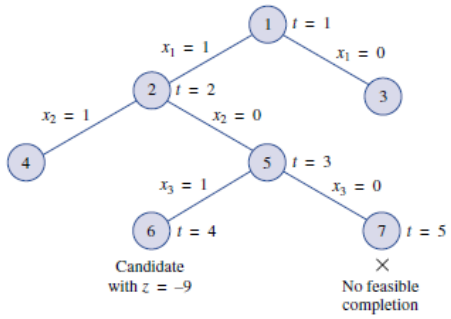
$$z_{int} \leq z^*_{int} \leq z^*_{LP},$$

for a general maximization problem.

In the branch-and-bound search, we solve a sequence of **subproblems** which are *LP relaxations* of the IP.

• A display of all subproblems that have been created is called a **tree**.

• Each subproblem is referred to as a **node** of the tree.



Branch-and-Bound for IP problems

Branching is concerned with *generating new subproblems* at a node of the tree. The general philosophy for the branching part is as follows:

• Solve the LP relaxation (subproblem 1) of the IP.

• Choose a variable with a fractional optimal value x_j^*. Since x_j should be integer, it must satisfy one of the following two constraints:

$$x_j \leq \lfloor x_j^* \rfloor \quad or \quad x_j \geq \lceil x_j^* \rceil$$

• Solve the two LP subproblems resulting from appending each of the branching constraints.

Bounding is concerned with *deciding to stop or not the branching process* for a given node. Suppose a *maximization* problem is considered.

• Any existing feasible integer solution that has been obtained provides a lower bound for the optimal IP objective value.

• Any further subproblems generated by appending additional branching constraints will have smaller objective values than that of the current node.

Therefore, *any problems in the tree search whose relaxations have optimum objective values (integral or not) which are smaller than the lower bound can be discarded.*

Pruning Criteria of the search tree:

• ***Value dominance***: the optimal objective value for the subproblem does not exceed (in a max problem) the current **lower bound**.

• ***Integrality***: the subproblem yields an optimal solution in which all variables have integer values.

• ***Infeasibility***: the subproblem is infeasible.

Procedure: Branch-and-Bound Algorithm (for max)
Step 0: Initialisation: Set $UB = \infty$ and $LB = -\infty$.
Step 1: Solve the LP relaxation of the IP at node 0. Set $UB = z_0^*$.
Step 2: If $UB = LB$ then **stop**; the optimal solution is the feasible solution with $z^* = LB$. Otherwise go to step 3.
Step 3: Branch on node:
Step 3.1: Select the node i with greatest z_i^*.
Step 3.2: Choose the variable x_j with a fractional value nearest to 1/2.

Step 3.3: Create two branches and two descendent nodes, one by appending $x_j \leq \lfloor x_j^* \rfloor$ and the other $x_j \geq \lceil x_j^* \rceil$.

Step 3.4: Solve the LP at each of the descendent nodes and *update UB* as the maximum z^* over all nodes from which there are no branches; *update LB* as the maximum z^*over integer feasible solutions.

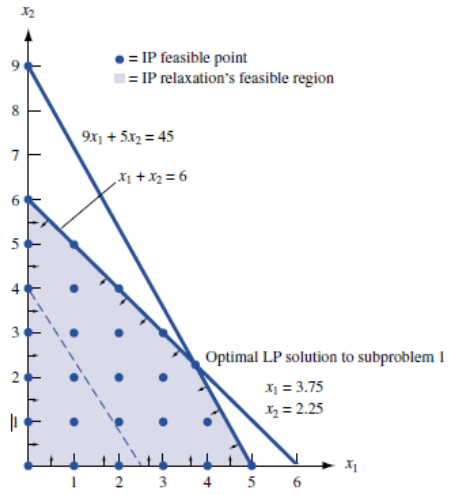
Step 3.5: If at any node i we have $z^* < LB$, discard it from further branching steps.

Step 3.6: Go to step 2.

Example

Solve the following IP by Branch-and-Bound:

$$\begin{aligned} max \quad & 8x_1 + 5x_2 \\ s.t. \quad & x_1 + x_2 \leq 6 \\ & 9x_1 + 5x_2 \leq 45 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$



Solution:

Problem1:

Considering the linear relaxation and adding slack variables $x_3, x_4 \geq 0$, our initial tableau is

	x_1	x_2	x_3	x_4	
z	-8	-5	0	0	0
x_3	1	1	1	0	6
x_4	9	5	0	1	45

Pivoting on 9, we have

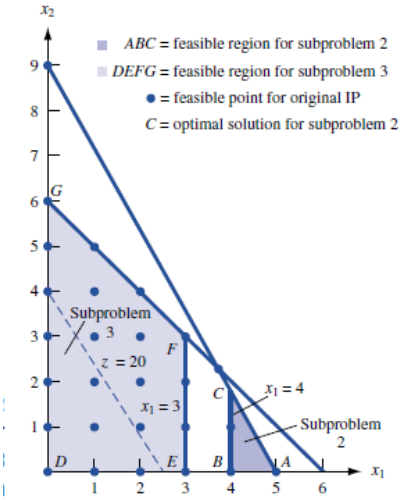
	x_1	x_2	x_3	x_4	
z	0	-5/9	0	8/9	40
x_3	0	**4/9**	1	-1/9	1
x_1	1	5/9	0	1/9	5

Pivoting on 4/9, we get

	x_1	x_2	x_3	x_4	
z	0	0	5/9	3/4	165/4
x_2	0	1	9/4	-1/4	9/4
x_1	1	0	-5/4	1/4	15/4

and this is optimal tableau with $x_2 = 9/4$, $x_1 = 15/4$. The optimal solution of LP relaxation is 165/4 = 41.25 and is an upper bound to the optimal of IP, since we are maximizing. Set $-\infty$ to be a lower bound of the problem.

We branch on $x_1 = 15/4$. Create two sub-problems with constraints $x_1 \leq \left\lfloor \frac{15}{4} \right\rfloor$ and $x_1 \geq \left\lceil \frac{15}{4} \right\rceil$ and let's consider



Subprobelm3:

The subproblem with constraint $x_1 \leq 3$. Adding a slack variable s_1, we have $x_1 + s_1 = 3$ and writing x_1 in terms of non-basic variables from the optimal tableau (final tableau of subproblem1), we have

$$x_1 - \frac{5}{4}x_3 + \frac{1}{4}x_4 = \frac{15}{4}$$

$x_1 = \frac{5}{4}x_3 - \frac{1}{4}x_4 + \frac{15}{4}$ Substitute in $x_1 = -s_1 + 3$ then

$$\frac{5}{4}x_3 - \frac{1}{4}x_4 + s_1 = -\frac{3}{4}$$

Now, adding the variable s_1 to our tableau and reoptimizing by **dual simplex**,

	x_1	x_2	x_3	x_4	s_1	
z	0	0	5/9	3/4	0	165/4
x_2	0	1	9/4	-1/4	0	9/4
x_1	1	0	-5/4	1/4	0	15/4
s_1	0	0	5/4	**-1/4**	1	-3/4

	x_1	x_2	x_3	x_4	s_1	
z	0	0	155/36	0	3	39
x_2	0	1	-1	0	-1	3
x_1	1	0	-5/2	0	1	3
x_4	0	0	-5	1	-4	3

By this tableau, we have found integer solution (3, 3), with z = 39 (new LB because $x_1 \leq 3$) . Update the **lower bound** to 39 and this node (subproblem) is pruned by integrality.

Subproblem2:
Consider the subproblem for $x_1 \geq 4$. Introducing a surplus variable s_2 we get $x_1 - s_2 = 4$ and expressing x_1 in terms of non-basic variables from (final tableau of subproblem1) $x_1 = \frac{5}{4}x_3 - \frac{1}{4}x_4 + \frac{15}{4}$ Substitute in $x_1 - s_2 = 4$ we get $\frac{5}{4}x_3 - \frac{1}{4}x_4 + \frac{15}{4} - s_2 = 4$ then
$(\frac{5}{4}x_3 - \frac{1}{4}x_4 - s_2 = \frac{16-15}{4} = \frac{1}{4}$) multiply by -1, we get $-\frac{5}{4}x_3 + \frac{1}{4}x_4 + s_2 = -\frac{1}{4}$.

Adding this constraint to the optimal tableau of the LP,

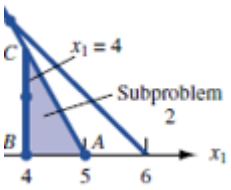
	x_1	x_2	x_3	x_4	s_2	
z	0	0	5/9	3/4	0	165/4
x_2	0	1	9/4	-1/4	0	9/4
x_1	1	0	-5/4	1/4	0	15/4
s_2	0	0	**-5/4**	1/4	1	-1/4

after one iteration

	x_1	x_2	x_3	x_4	s_2	
z	0	0	0	1	1	41
x_2	0	1	0	1/5	9/5	9/5
x_1	1	0	0	0	-1	4
x_3	0	0	1	-1/5	-4/5	1/5

yielding optimal solution (4, 9/5), with objective 41 (new UB because $x_1 \geq 4$). The upper and lower bounds of our IP problem are now 41 and 39. We branch on $x_2 = \frac{9}{5} = 1.8$ and we have the constraints $x_2 \leq 1$ and $x_2 \geq 2$.

Subproblem4: سوف نعمل في منطقة المسالة 2 اعلاه



Let's consider the second constraint $x_2 \geq 2$. Our problem is

$$\begin{aligned} max \quad & 8x_1 + 5x_2 \\ s.t. \quad & x_1 + x_2 \leq 6 \\ & 9x_1 + 5x_2 \leq 45 \\ & x_1 \geq 4 \quad \text{constraint from subp3} \\ & x_2 \geq 2 \quad \text{we add new branch} \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

and note that for $x_1 = 4$ and $x_2 = 2$, $9x_1 + 5x_2$ is violated, thus this problem is **infeasible** and its node is pruned by infeasibility.

Subproblem5:
Consider now the constraint $x_2 \leq 1$.

$$\begin{aligned} max \quad & 8x_1 + 5x_2 \\ s.t. \quad & x_1 + x_2 \leq 6 \\ & 9x_1 + 5x_2 \leq 45 \\ & x_1 \geq 4 \quad \text{constraint from subp3} \\ & x_2 \leq 1 \quad \text{we add new branch} \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

Adding a slack variable s_3, we have that $x_2 + s_3 = 1$ and in terms of non-basic variables we can write

$$x_2 = \frac{9}{5} - \frac{1}{5}x_4 - \frac{9}{5}s_2.$$

Substituting,

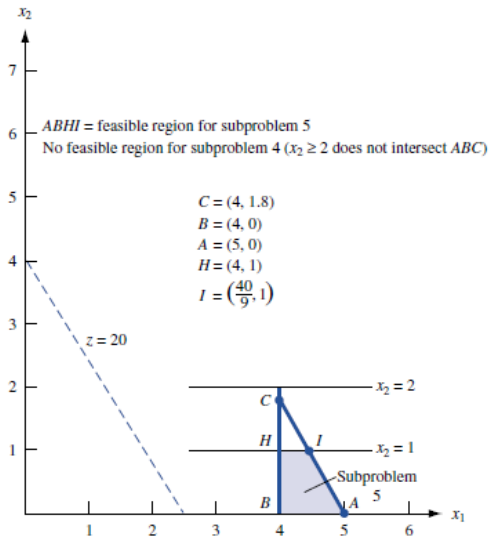
$$\boldsymbol{s_3 - \frac{1}{5}x_4 - \frac{9}{5}s_2 = -\frac{4}{5}}$$

and adding this constraint to the previous tableau and solve it by dual simplex

	x_1	x_2	x_3	x_4	s_2	s_3	
z	0	0	0	1	1	0	41
x_2	0	1	0	1/5	9/5	0	9/5
x_1	1	0	0	0	-1	0	4
x_3	0	0	1	-1/5	-4/5	0	1/5
s_3	0	0	0	-1/5	**-9/5**	1	-4/5

	x_1	x_2	x_3	x_4	s_2	s_3	
z	0	0	0	8/9	0	5/9	365/9
x_2	0	1	0	0	0	1	1
x_1	1	0	0	1/9	0	-5/9	40/9
x_3	0	0	1	-1/9	0	-4/9	5/9
s_2	0	0	0	1/9	1	-5/9	4/9

and optimal solution is (40/9, 1) = (4.44, 1) with objective 365/9. The upper bound now becomes equal to 365/9=40.55.



Branch on $x_1 = \frac{40}{9} = 4.4$ and we obtain that $x_1 \leq 4$ and $x_1 \geq 5$.

Subproblem7:

Take $x_1 \leq 4$ and add a slack variable s_4, so to have $x_1 + s_4 = 4$ so the constraint is $-\frac{1}{9}x_4 + \frac{5}{9}s_3 + s_4 = -\frac{4}{9}$ and the updated tableau is above

	x_1	x_2	x_3	x_4	s_2	s_3	s_4	
z	0	0	0	8/9	0	5/9	0	365/9
x_2	0	1	0	0	0	1	0	1
x_1	1	0	0	1/9	0	-5/9	0	40/9
x_3	0	0	1	-1/9	0	-4/9	0	5/9
s_2	0	0	0	1/9	1	-5/9	0	4/9
s_4	0	0	0	**-1/9**	0	5/9	1	-4/9

and immediately we obtain integer solution (4, 1) with objective 37(which is lower). This solution is dominated (is less in case of max problems) by the earlier found solution (3, 3) with objective 39 (which was a lower bound) and thus is pruned by value dominance.

Subproblem6:

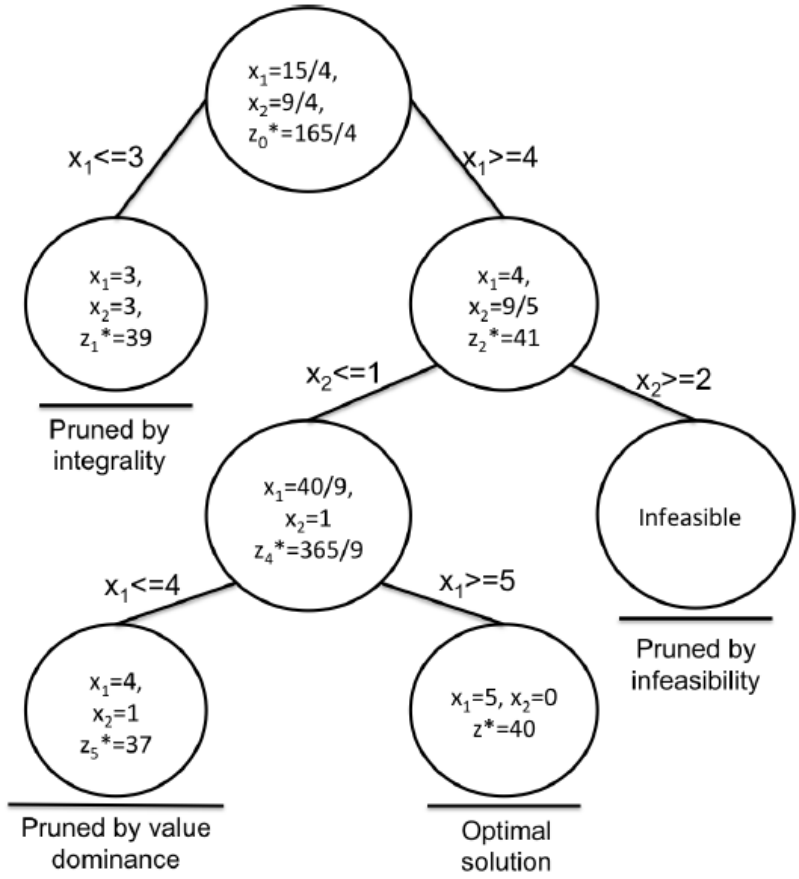
We consider the case $x_1 \geq 5$. Subtracting a surplus variable s_5, $x_1 - s_5 = 5$ and rewriting as before x_1 in terms of non-basic variables, we have that

$$\frac{1}{9}x_4 - \frac{5}{9}s_3 + s_5 = -\frac{5}{9}$$

	x_1	x_2	x_3	x_4	s_2	s_3	s_5	
z	0	0	0	8/9	0	5/9	0	365/9
x_2	0	1	0	0	0	1	0	1
x_1	1	0	0	1/9	0	-5/9	0	40/9
x_3	0	0	1	-1/9	0	-4/9	0	5/9
s_2	0	0	0	1/9	1	-5/9	0	4/9
s_5	0	0	0	1/9	0	**-5/9**	1	-5/9

Subtracting the last row from the row of basic variable x_1 and adding 5/9 times the second row to last row we find integer solution (5, 0), with objective 40 which dominates the solution (3, 3) with objective 39. Thus, **we set our upper and lower bound equal to 40 and we are done.** The optimal solution of our IP program is (5, 0) with maximum value of objective function equal to 40.

Below, we depict by a binary tree the operation of Branch & Bound algorithm to the problem.



Omit LB = 39 by integrality,
Omit LB = 37 by dominance (39>37),
Omit x2>=2 by infeasible,

Remark

• The Branch-and-Bound algorithm can be easily extended to Mixed-integer problems. The idea is to branch only on those variables that are required to be integer.

H.W:

Use branch-and-bound to solve the following IPs:

1 $\max z = 5x_1 + 2x_2$
s.t. $3x_1 + x_2 \leq 12$
$x_1 + x_2 \leq 5$
$x_1, x_2 \geq 0; x_1, x_2$ integer